A Perspective On Amalgamated Rings Via Symmetricity

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Motivation

Throughout this work all rings are associative with identity unless otherwise stated. Let A and B be commutative rings with a ring homomorphism $f : A \to B$ and I be an ideal of B. The amalgamation of A with B along an ideal I of B with respect to f (denoted by $A \bowtie^f I$) introduced and studied by D'Anna and Fontana-2007.

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In this setting, we consider the following subring of $A \times B$ (endowed with the usual componentwise operations):

$$A \bowtie^f I := \{(a, f(a) + i) \mid a \in A, i \in I\}$$

is called *amalgamated construction of A with B along I with respect to f*.

The authors studied characterizations for $A \bowtie^f I$ to be a Noetherian ring, an integral domain, a reduced ring and they characterized those distinguished pullbacks that can be expressed as an amalgamation provided the rings are commutative. Also ideal extensions are defined and investigated for noncommutative rings by Nicholson and Zhou-2005. Clean properties of amalgamated rings in commutative case are studied Chhiti-Mahdou-Tamekkante. Some homological properties of amalgamated duplication of a ring along an ideal are investigated by Chhiti and Mahdou. Bezout properties of amalgamated rings are studied by Kabbour and Mahdou.

Abstract

Let A and B be two rings (not necessarily commutative) with identity, I an ideal of B and $f : A \to B$ a ring homomorphism. In this work, we deal with some versions of reversibility and symmetricity on amalgamated rings along an ideal. This work aims at studying the transfer of the notion of reversible rings, weakly reversible rings, symmetric rings, weak symmetric rings to the amalgamation of rings along ideals.

Reversible ring

A ring R is called *reversible* if for any $a, b \in R$, $ab = 0$ implies $ba = 0$. This ring is introduced and studied by Cohn, 1999.

We investigate the conditions on the reversibility of the rings of the form $A \bowtie^f I$. We start with some examples to illustrate the definition.

Example

Let $A = \mathbb{Z}_2$ and $B = \begin{bmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{bmatrix}$ be the rings and $I = \begin{bmatrix} 0 & \mathbb{Z}_2 \\ 0 & 0 \end{bmatrix}$ be the ideal of B and $f : A \to B$ defined by $f(a) = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ where $a \in \mathbb{Z}_2$. A is a reversible ring, B is not reversible. For $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \in B$ $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = 0$ but $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \neq 0$. $f(A) + I$ is reversible. And

$$A \bowtie^f I = \left\{ (0, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}), (0, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}), (1, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}), (1, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}) \right\}$$

is reversible.

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Proof: Let a, $b \in A$ with $ab = 0$. Then $(a, f(a))(b, f(b)) = 0$ in $A \bowtie^f I$. By hypothesis $(b, f(b))(a, f(a)) = 0$. So $ba = 0$.

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(2) If A and $f(A) + I$ are reversible, so is $A \bowtie^f I$.

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(2) If A and $f(A) + I$ are reversible, so is $A \bowtie^f I$.
Proof: Let $(a, f(a) + x), (b, f(b) + y) \in A \bowtie^f I$ with $(a, f(a) + x)(b, f(b) + y) = 0$. Then $ab = 0$ and $(f(a) + x)(f(b) + y) = 0$. By hypothesis, $ba = 0$ and $(f(b) + y)(f(a) + x) = 0$. It follows that $A \bowtie^f I$ is reversible.

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Proof: We first show that A is reversible. Let a, $b \in A$ with $ab = 0$. Then $f(a) + 0, f(b) + 0 \in f(A) + I$ and $(f(a) + 0)(f(b) + 0) = f(ab) = 0$. By hypothesis, $(f(b) + 0)(f(a) + 0) = 0 = f(ba)$. By the injectivity of f, $ba = 0$. So A is reversible. By (2) $A \bowtie^f I$ is reversible.

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(b) If $f(A) + I$ is reversible, then $A \bowtie^f I$ is reversible.

Proof: (a) Note that $f(A) + I$ is reversible as a subring of the reversible ring B. It is clear that $f(A) + I$ is isomorphic to $A \bowtie^f I$ by the homomorphism α defined by $\alpha(a, f(a) + x) = f(a) + x$ where $(a, f(a) + x) \in A \bowtie^f I$.
(b) Clear by the property that reversibility is preserved under isomorphism.

Example

Let $A = \mathbb{Z}_2$ and $B = \begin{bmatrix} \mathbb{Z}_2 & 0 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 & 0 \\ 0 & 0 & \mathbb{Z}_2 \end{bmatrix}$ be the rings and

$I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \mathbb{Z}_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ be the ideal of B and $f : A \to B$ defined by

$f(a) = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$. A is reversible and B is not reversible.

Also f is injective and

$$f(A)+I=\left\{\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix},\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix},\begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}\right\}$$

is reversible. By Theorem, $A\bowtie^f I$ is a reversible ring.

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Lemma

The following hold.

(1) If the rings A and B are reversible, then $A \bowtie^f I$ is reversible.

(2) Let $B = A$, $f = id_A$ and $I = (0)$. In this situation, $A \bowtie^f I$ is reversible if and only if A and B are reversible.

(3) If $A \bowtie^f I$ is reversible, then every subring is reversible, in particular, the ideals $(0) \bowtie^f I$, $A \bowtie^f (0)$ and $I_1 \bowtie^f I$ are reversible where I_1 is any ideal of A.

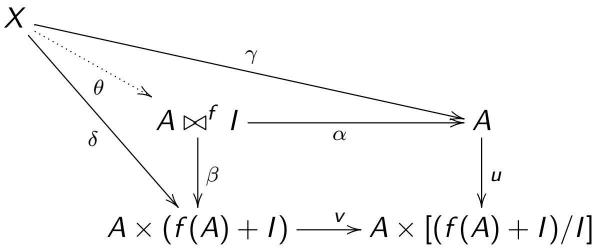
Definition

We recall that, if $\alpha : A \to C$, $\beta : B \to C$ are ring homomorphisms, the subring $D := \alpha \times_C \beta := \{(a, b) \in A \times B \mid \alpha(a) = \beta(b)\}$ of $A \times B$ is called the *pullback*(or *fiber product*) of α and β (D'Anna, Finocchiaro, Fontana-2007).

Theorem

Let A and B be rings and I be an ideal of B and $f : A \to B$ a homomorphism. Let $u : A \to A \times \big((f(A)+I)/I\big)$ defined by $u(a) = (a, f(a)+I)$ and $v : A \times (f(A)+I) \to A \times \big((f(A)+I)/I\big)$ defined by $v(a, f(b)+x) = (a, f(b)+I)$. Then $A \bowtie^f I$ is the pullback of the maps u and v.

Proof: Consider the following diagram with $\alpha(a, f(a) + t) = a$ and $\beta(a, f(a) + t) = (a, f(a) + t) \in A \times (f(A) + I)$ where $(a, f(a) + t) \in A \bowtie^f I$.



Then $u\alpha = v\beta$. For if $(a, f(a)+t) \in A \bowtie^f I$, then $u\alpha(a, f(a)+t) = u(a) = (a, f(a)+I)$, $v\beta(a, f(a)+t) = (a, f(a)+I)$. Let X be any module and γ and δ homomorphisms such that $v\delta = u\gamma$. For any $x \in X$, set $\gamma(x) = a \in A$. Then $u(a) = (a, f(a)+I)$ and $\delta(x) = (b, f(c)+i)$ for some b, $c \in A$ and $i \in I$. $v\delta = u\gamma$ implies $a = b$ and $f(c) - f(a) \in I$. Define $\theta(x) = (a, f(a)+i)$. Then $\beta\theta = \delta$ and $\alpha\theta = \gamma$. θ is unique, for if ν satisfies $\alpha\nu = \gamma$ and $\beta\nu = \delta$, then $\nu = \theta$ since β is a monomorphism. This completes the proof.

Proposition

With the notation of Definition, we have:

(1) If A is reversible and β is injective, then D is reversible.

Proposition

With the notation of Definition, we have:

(1) If A is reversible and β is injective, then D is reversible. Proof: Assume that A is reversible. Let (a, b) and (c, d) be in D with $(a, b)(c, d) = 0$, then $ac = 0$, $bd = 0$. We have $ca = 0$ and $\alpha(ca) = \alpha(c)\alpha(a) = \beta(d)\beta(b) = \beta(db) = 0$, then $db = 0$ since β is injective. So D is reversible.

(2) If at least one of the following conditions holds

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then D is reversible.

Proof: By the symmetry of conditions (a) and (b), it is enough to show that condition (a) holds. Let (a, b) and (c, d) be in D with $(a, b)(c, d) = 0$. Then $ac = 0$ and $bd = 0$. Since A is reversible, we have $ca = 0$. Also, $\beta(db) = \beta(d)\beta(b) = \alpha(c)\alpha(a) = 0$, so $db \in Ker\beta$. Also $(db)^2 = dbdb = 0$, so $db \in nil(B) \cap Ker\beta = \{0\}$. Therefore D is reversible.

Weakly Reversible Rings

Reversible rings are generalized by Zhao-Yang, a ring R is called *weakly reversible* if for all $a, b, r \in R$ such that $ab = 0$, $Rbra$ is a nil left ideal of R(equivalently, $braR$ is a nil right ideal of R). Weakly reversible rings are also studied in Kose et al. Reversible rings are weakly reversible. There are weakly reversible rings that are not reversible as the following example shows.

Example

Let $A = \mathbb{Z}_2$ and $B = \begin{bmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{bmatrix}$ be the rings and $I = \begin{bmatrix} 0 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{bmatrix}$ the ideal of B and $f : A \to B$ defined by $f(a) = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ where $a \in \mathbb{Z}_2$. Then f is injective $f(A) \cap I = 0$ and A is weakly reversible. By Zhao-Yang [Proposition 2.3, 2007], B is weakly reversible. It is easy to check that $A \bowtie^f I$ is weakly reversible. Now $(0, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix})(1, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}) = 0$ in $A \bowtie^f I$, but $(1, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix})(0, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}) \neq 0$. So $A \bowtie^f I$ is not reversible.

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Let A and B be a pair of rings, $f : A \to B$ be a ring homomorphism and I be a proper ideal of B.

(1) Assume that f is injective. If $A \bowtie^f I$ is weakly reversible and $f(A) \cap I = \{0\}$, then both A and $f(A) + I$ are weakly reversible.

(2) If A and $f(A) + I$ are weakly reversible, so is $A \bowtie^f I$.

(3) Assume that f is injective. If $f(A) + I$ is weakly reversible, then so are A and $A \bowtie^f I$.

(4) Let $f^{-1}(I) = \{0\}$.
   (a) If B is weakly reversible, then $A \bowtie^f I$ is weakly reversible.
   (b) If $f(A) + I$ is weakly reversible, then $A \bowtie^f I$ is weakly reversible.

Symmetricity Of Amalgamated Rings

Symmetric rings are defined by Lambek in 1971. A ring R is called *symmetric* if $abc = 0$ implies $acb = 0$ for a, b, $c \in R$. Clearly symmetric property of rings are preserved under isomorphisms and under subrings. In this section we study necessary and sufficient conditions for $A \bowtie^f I$ to be symmetric.

Theorem

Let A and B be a pair of rings, $f : A \to B$ be a ring homomorphism and I be a proper ideal of B.

(1) If $A \bowtie^f I$ is symmetric, then so is A.

(2) If A and $f(A) + I$ are symmetric, then so is $A \bowtie^f I$.

(3) Assume that $I \cap S \neq \emptyset$ where S is the set of regular central elements of B. Then $A \bowtie^f I$ is a symmetric ring if and only if $f(A) + I$ and A are symmetric rings.

(4) Assume that f is injective and $f(A) + I$ is a symmetric ring. Then $A \bowtie^f I$ is a symmetric ring.

(5) Assume that $f^{-1}(I) = \{0\}$. If $f(A) + I$ is a symmetric ring, then $A \bowtie^f I$ is a symmetric ring.

Weak Symmetricity Of Amalgamated Rings

In this section we study weak symmetric rings. Ouyang and Chen discussed weak symmetric rings. A ring R is called *weak symmetric* if $abc \in \text{nil}(R)$ implies $acb \in \text{nil}(R)$ for all a, b, $c \in R$. It is proved that all symmetric rings are weak symmetric. Clearly weak symmetric property of rings are preserved under isomorphisms and under subrings. In this section we study necessary and sufficient conditions for $A \bowtie^f I$ to be weak symmetric.

Theorem

Let A and B be a pair of rings, $f : A \to B$ be a ring homomorphism and I be a proper ideal of B.

(1) If $A \bowtie^f I$ is weak symmetric, then so is A.

(2) If A and $f(A) + I$ are weak symmetric, then so is $A \bowtie^f I$.

(3) Assume that $I \cap S \neq \emptyset$ where S is the set of regular central elements of B. Then $A \bowtie^f I$ is weak symmetric if and only if $f(A) + I$ and A are weak symmetric.

(4) Assume that $f^{-1}(I) \subseteq \text{nil}(A)$. If $f(A) + I$ is a weak symmetric ring, then $A \bowtie^f I$ is weak symmetric.

(5) If $f(A) + I$ is weak symmetric and f is injective, then A and I are weak symmetric.

Thank you very much for your attention!